# Introduction to Partial Differential Equations
## Exercise sheet 1

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This exercise sheet is revision and does not count towards the exercise points. But please feel free to attempt them (before or after the tutorial) and submit them for correction any way.

1. **Chain rule in multiple variables** Recall the chain rule for functions of multivariable variables (Satz 10.4(iii) in Schmidt's Analysis II script): Let $f : U \subset X \to Y$ be differentiable at $x_0 \in U$ and $g : V \subset Y \to Z$ be differentiable at $f(x_0) \in f[U] \subset V$. Then $g \circ f$ is differentiable at $x_0$ and

$$(g \circ f)'(x_0) = g'(f(x_0)) \circ f'(x_0).$$

   **(a)** Why does this chain rule above use function composition, when the chain rule for functions of a single variable uses multiplication? i.e.

$$\frac{d}{dx}(x^2+1)^3 = 3(x^2+1)^2 \cdot 2x. = 6x(x^2+1)^2.$$

   **(b)** Suppose that $u : \mathbb{R}^n \to \mathbb{R}$ and $x : \mathbb{R} \to \mathbb{R}^n$. Express the chain rule with partial derivatives to show that

$$\frac{d}{dt}u(x(t)) = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{dx_i}{dt}.$$

   **(c)** Write the above formula in terms of gradients and dot products.

   **(d)** Consider the function $u(x, y) = x^2 + 2y$ and the polar coordinates $x = r\cos\theta, y = r\sin\theta$. Compute the radial and angular derivatives of $u$.

   **(e)** Consider a scalar function $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ of $2n+1$ variables and a function $u : \mathbb{R}^n \to \mathbb{R}$. Write an expression for the derivative of $F(\nabla u(x), u(x), x)$ with respect to $x_1$.

2. **Contour Diagrams**

   Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^2 + y\,(x + 1) - y^3$.

   **(a)** Use computer assistance to draw a *contour diagram* for this function. For example https://www.desmos.com/calculator/8equb62lyq. A contour is another word for a level set $f^{-1}[\{c\}]$.

   **(b)** What is the maximum and minimum of this function? Where are its critical points? (Give approximate values.)

3. **Multiindices and the Generalised Leibniz rule** In this question we introduce multiindex notation. A *multiindex* of $n$ variables is a vector $\gamma \in \mathbb{N}_0^n$.

   **(a)** Let $x = (x_1, x_2, x_3)$ be coordinates on $\mathbb{R}^3$. Write out the full expression for the derivative $\partial^{(0,2,1)}$.

   **(b)** Why do we need to assume that partial derivatives commute for multiindex notation to be useful?

   **(c)** Which multiindices satisfy $|\gamma| \leq 2$ and which satisfy $\gamma \leq (0, 2, 1)$?